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Existence results for extended vector variational-like inequality

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Abstract In the present work, we introduce and study an extended vector variational-like inequality in Banach spaces. Some existence results for extended vector variational-like inequality are obtained by using g–h–η-quasimonotone of Stampacchia and Minty types mappings.

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1. Introduction and preliminaries

In 1980, Giannessi [1] introduced vector variational inequality in a finite dimensional Euclidean space. The vector variational inequality is a generalized form of a variational inequality, having applications in different areas of operations research, economics equilibrium, optimal control, free boundary value problems and optimization. In 2007, Zhao and Xia [2] obtained some existence results for vector variational-like inequality by using the definition of properly η-quasimonotone of Stampacchia and Minty types. In 2010, Irfan and Ahmad [3] introduced and studied generalized multivalued vector variational-like inequality by using the concept of escaping sequences. For details, we can refer to [4–8] and reference therein.

In this paper, we introduce and study an extended vector variational-like inequality in Banach spaces. Some existence results for extended vector variational-like inequality are obtained by using g–h–η-quasimonotone of Stampacchia and Minty types mappings.

Let X and Y be two real Banach spaces, $K \subset X$ be a nonempty, closed and convex subset, $C \subset Y$ be a pointed, closed and convex cone in Y such that $\mathrm{int}C \neq \phi$ where $\mathrm{int}C$ denote the interior of C and let $L(X, Y)$ be the space of all continuous linear mappings from X to Y. For any $l \in L(X, Y), x \in X$, let $\langle l, x\rangle$ denote the value of l at x. Let $S, T: K \to L(X, Y)$, $g: K \to K, \eta: K \times K \to X$ and $h: K \times K \to Y$ are mappings. We consider the following *extended vector variational-like iequalities*:

$$\text{(EVVLI-I)} \quad \begin{cases} \text{Find } x \in K \text{ such that,} \\ \langle S(x) + T(x), \eta(g(y), g(x))\rangle + h(g(y), g(x)) \geqslant_C 0, \\ \quad \forall y \in K, \end{cases}$$

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and

$$\text{(EVVLI-II)}\ \begin{cases} \text{Find } x \in K \text{ such that,} \\ \langle S(y)+T(y), \eta(g(x),g(y))\rangle + h(g(x),g(y)) \leqslant_C 0, \\ \quad \forall y \in K. \end{cases}$$

Special cases:

(i) If $S \equiv 0$, and $g = I$ then (EVVLI-I) and (EVVLI-II) reduces to the following *vector variational-like inequalities* considered and studied by Ahmad [9]

$$\text{(VVLI-I)}\quad \begin{cases} \text{Find } x \in K \text{ such that,} \\ \langle T(x), \eta(y,x)\rangle + h(y,x) \geqslant_C 0, \quad \forall y \in K, \end{cases}$$

and

$$\text{(VVLI-II)}\quad \begin{cases} \text{Find } x \in K \text{ such that,} \\ \langle T(y), \eta(x,y)\rangle + h(x,y) \leqslant_C 0, \quad \forall y \in K. \end{cases}$$

(ii) If $S, h \equiv 0$ and $g = I$ then (EVVLI-I) and (EVVLI-II) reduces to the following *vector variational-like inequalities* considered and studied by Zhao and Xia [2]

$$\text{(VVLI-I)}\quad \begin{cases} \text{Find } x \in K \text{ such that,} \\ \langle T(x), \eta(y,x)\rangle \geqslant_C 0, \quad \forall y \in K, \end{cases}$$

and

$$\text{(VVLI-II)}\quad \begin{cases} \text{Find } x \in K \text{ such that,} \\ \langle T(y), \eta(x,y)\rangle \leqslant_C 0, \quad \forall y \in K. \end{cases}$$

The following concepts and results are needed for the results.

Definition 1.1. A mapping $f:\ K \to Y$ is said to be *hemicontinuous* if, for any fixed $x, y \in K$, the mapping $t \mapsto f(x + t(y - x))$ is continuous at 0^+.

Definition 1.2. Let $C:\ K \to 2^Y$ be a set-valued mapping, $h:\ K \times K \to Y$ and $\eta:\ K \times K \to X$ are the single-valued mappings. Then

(i) $h(\cdot, v)$ is said to be *C-convex* in the first argument if

$$h(tu_1 + (1-t)u_2, v) \in th(u_1, v) + (1-t)h(u_2, v) - C, \quad \forall u_1, u_2 \in K, \quad t \in [0, 1].$$

(ii) If K is an affine set, the $\eta(x, y)$ is said to be *affine* with respect to u if for any given $v \in K$

$$\eta(tu_1 + (1-t)u_2, v) = t\eta(u_1, v) + (1-t)\eta(u_2, v) - C, \quad \forall u_1, u_2 \in K, \quad t \in \mathbb{R}$$

with $u = tu_1 + (1-t)u_2 \in K$.

Definition 1.3. Let $S, T:\ K \to L(X, Y)$, $\eta:\ K \times K \to X$, $h:\ K \times K \to Y$ and $g:\ K \to K$ be mappings. Then S and T are said to be g–h–η-pseudomonotone if for any $x, y \in K$,

$$\langle S(x) + T(x), \eta(g(y), g(x))\rangle + h(g(y), g(x)) \geqslant_C 0,$$

$$\Rightarrow \langle S(y) + T(y), \eta(g(x), g(y))\rangle + h(g(x), g(y)) \leqslant_C 0.$$

Example 1.1. Let $X = \mathbb{R}, K = \mathbb{R}_+, Y = \mathbb{R}^2, C = \mathbb{R}^2_+$ and $S(x) = \begin{pmatrix} 3 + \sin x \\ 3 + \cos x \end{pmatrix}$, $T(x) = \begin{pmatrix} \cos x + 2 \\ \sin x + 2 \end{pmatrix}$, $g(x) = 2x, \eta(y, x) = 3y - 4x$ and $h(x, y) = \begin{pmatrix} 6y - 8x \\ 3y^2 - xy - 4x^2 \end{pmatrix}$, $\forall x, y \in K$.

Then $\forall x, y \in K$

$$\begin{aligned} &\langle S(x) + T(x), \eta(g(y), g(x))\rangle + h(g(y), g(x)) \\ &\quad = \begin{pmatrix} 5 + 2\sin x \\ 5 + 2\cos x \end{pmatrix}(3y - 4x) + \begin{pmatrix} 12y - 16x \\ 12y^2 - 4xy - 16x^2 \end{pmatrix} \\ &\quad = \begin{pmatrix} (5 + 2\sin x)(3y - 4x) \\ (5 + 2\cos x)(3y - 4x) \end{pmatrix} + \begin{pmatrix} 4(3y - 4x) \\ 4(3y - 4x)(y + x) \end{pmatrix} \\ &\quad = (3y - 4x)\begin{pmatrix} 9 + 2\sin x \\ 5 + 2\cos x + 4y + 4x \end{pmatrix} \geqslant_C 0, \end{aligned}$$

implies that $3y > 4x$. So it follows that

$$\begin{aligned} &\langle S(y) + T(y), \eta(g(x), g(y))\rangle + h(g(x), g(y)) \\ &\quad = \begin{pmatrix} 5 + 2\sin x \\ 5 + 2\cos x \end{pmatrix}(3x - 4y) + \begin{pmatrix} 12x - 16y \\ 12x^2 - 4xy - 16y^2 \end{pmatrix} \\ &\quad = (3x - 4y)\begin{pmatrix} 9 + 2\sin y \\ 5 + 2\cos y + 4x + 4y \end{pmatrix} \leqslant_C 0. \end{aligned}$$

$\Rightarrow$ S and T are g–h–η-pseudomonotone.

Definition 1.4 [10]. A multi-valued operator $S:\ X \to 2^{X^*}$ is called *quasimonotone* if for all $x, y \in X$ the following implications hold:

$$\exists x^* \in S(x):\ \langle x^*, y - x\rangle > 0 \Rightarrow \exists y^* \in S(y):\ \langle y^*, y - x\rangle \geqslant 0.$$

Definition 1.5 [10]. A multi-valued operator $S:\ X \to 2^{X^*}$ is called *properly quasimonotone* if for every $x_1, x_2, \ldots, x_n \in X$ and every $y \in \mathrm{Conv}\{x_1, x_2, \ldots, x_n\}$ there exist i such that

$$\forall x_i^* \in S(x_i):\ \langle x_i^*, y - x_i\rangle \leqslant 0.$$

Definition 1.6. A mapping $T:\ K \to L(X, Y)$ is said to be *properly quasimonotone of Stampacchia type* if for all $n \in N$ for all vectors $v_1, v_2, \ldots, v_n \in K$ and scalars $\lambda_i \geqslant 0, i = 1, 2, \ldots, n$ with $\sum_{i=1}^n \lambda_i = 1$ and $u := \sum_{i=1}^n \lambda_i v_i, \langle Tu, v_i - u\rangle \geqslant_C 0$ holds for some i. T is said to be *properly quasimonotone of Minty type* if for all vectors $v_1, v_2, \ldots, v_n \in K$ and scalars $\lambda_i \geqslant 0$, $i = 1, 2, \ldots, n$ with $\sum_{i=1}^n \lambda_i = 1$ and $u := \sum_{i=1}^n \lambda_i v_i, \langle Tv_i, v_i - u\rangle \geqslant_C 0$ holds for some i.

Definition 1.7. A mapping $T:\ K \to L(X, Y)$, $\eta:\ K \times K \to K$ and $g:\ K \to K$ be three mappings. T is said to be *properly g–η-quasimonotone of Stampacchia type* if for all $n \in N$, for all vectors $v_1, v_2, \ldots, v_n \in K$ and scalars $\lambda_i \geqslant 0, i = 1, 2, \ldots, n$ with $\sum_{i=1}^n \lambda_i = 1$ and $u := \sum_{i=1}^n \lambda_i v_i, \langle Tu, \eta(g(v_i), g(u))\rangle \geqslant_C 0$ holds for some i. T is said to be *properly g–η-quasimonotone of Minty type* if for all vectors $v_1, v_2, \ldots, v_n \in K$ and scalars $\lambda_i \geqslant 0, i = 1, 2, \ldots, n$ with $\sum_{i=1}^n \lambda_i = 1$ and $u := \sum_{i=1}^n \lambda_i v_i, \langle Tv_i, \eta(g(v_i), g(u))\rangle \geqslant_C 0$ holds for some i.

Definition 1.8. A mapping $T:\ K \to L(X, Y)$, $\eta:\ K \times K \to X$, $h:\ K \times K \to Y$ and $g:\ K \to K$ be mappings. T is said to be *properly g–h–η-quasimonotone of Stampacchia type* if for all $n \in N$, for all vectors $v_1, v_2, \ldots, v_n \in K$ and scalars $\lambda_i \geqslant 0$,

$i=1,2,\ldots,n$ with $\sum_{i=1}^{n}\lambda_i=1$ and $u:=\sum_{i=1}^{n}\lambda_i v_i, \langle Tu,\eta(g(v_i),g(u))\rangle+h(g(v_i),g(u))\geqslant_C 0$ holds for some i. T is said to be *properly g–h–η-quasimonotone of Minty type* if for all vectors $v_1,v_2,\ldots,v_n\in K$ and scalars $\lambda_i\geqslant 0, i=1,2,\ldots,n$ with $\sum_{i=1}^{n}\lambda_i=1$ and $u:=\sum_{i=1}^{n}\lambda_i v_i, \langle Tv_i,\eta(g(u),g(v_i))\rangle+h(g(u),g(v_i))\leqslant_C 0$ holds for some i.

Example 1.2. Let X, KY, C be same as in Example 1.1 and $S(x)=\begin{pmatrix}6x^2\\9x^4\end{pmatrix}$, $T(x)=\begin{pmatrix}9x^2\\6x^4\end{pmatrix}$, $g(x)=3x, \eta(y,x)=2y-(x-3x^2)$ and $h(x,y)=\begin{pmatrix}y+x\\y^2+x^2\end{pmatrix}$, $\forall x,y\in K$.

We claim that S and T are properly g–h–η-quasimonotone of Stampacchia type. Suppose to the contrary that there exists $x_i\in K, t_i\geqslant 0, i=1,2,\ldots,n$ with $\sum_{i=1}^{n}t_i=1$ such that

$$\langle S(x)+T(x),\eta(g(x_i),g(x))\rangle+h(g(x_i),g(x))\ngeqslant_C 0,\quad i=1,2,\ldots,n,$$

where $x_i=\sum_{i=1}^{n}t_i x_i$, it follows that

$$\begin{aligned}\langle S(x)+T(x),\eta(g(x_i),g(x))\rangle+h(g(x_i),g(x)) &=\begin{pmatrix}15x^2\\15x^4\end{pmatrix}(6x_i-(3x-9x^2)+\begin{pmatrix}3x_i+3x\\4x_i^2+9x^2\end{pmatrix}\\ &=\begin{pmatrix}15x^2(6x_i-3x+9x^2)\\15x^4(6x_i-3x+9x^2)\end{pmatrix}+\begin{pmatrix}3x_i+3x)\\9x_i^2+9x^2\end{pmatrix}\ngeqslant_C 0\\ &=\begin{pmatrix}15x^2(6x_i-3x+9x^2)+3x_i+3x\\15x^4(6x_i-3x+9x^2)+9x_i^2+9x^2\end{pmatrix}\ngeqslant_C 0,\\ & i=1,\ldots,n,\end{aligned}$$

which is a contradiction, since

$$15x^2(6x_i-3x+9x^2)+3x_i+3x\geqslant_C 0,$$

and

$$15x^4(6x_i-3x+9x^2)+9x_i^2+9x^2\geqslant_C 0$$

for atleast one i. Thus S and T are properly g–h–η-quasimonotone of Stampacchia type.

Lemma 1.1. *Let $S,T: K\to L(X,Y)$, $\eta: K\times K\to X$, $h: K\times K\to Y$ and $g: K\to K$ be mappings. If S and T are g–h–η-pseudomonotone and properly g–h–η-quasimonotone of Stampacchia type, then S and T are properly g–h–η-quasimonotone of Minty type.*

Proof. The fact directly follows from Definitions 1.3 and 1.8. □

Definition 1.9. Let D be a nonempty subset of a topological Hausdorff space E. A mapping $G: D\to 2^E$ (the family of nonempty subset of E) is said to be a *KKM mapping* if for any finite set $\{x_1,x_2,\ldots,x_n\}\subset D$, $\mathrm{Co}\{x_1,x_2,\ldots,x_n\}\subset\bigcup_{i=n}^{n}G(x_i)$ where Co denotes the convex hull operator.

Lemma 1.2 [11]. *Let D be a nonempty subset of a topological Hausdorff vector space E and $G: D\to 2^E$ a KKM mapping. If $G(x)$ is closed for any $x\in D$ and compact for some $x\in D$, then $\bigcap_{x\in D}G(x)\neq\phi$.*

Lemma 1.3. *Let Y be a topological vector space with a pointed, closed and convex cone such that $intC\neq\phi$. Then for all $x,y,z\in Y$*

(*i*) *$x-y\in -C$ and $x\notin -intC\Rightarrow y\notin -intC$;*
(*ii*) *$x\in -intC$ and $y\notin -intC\Rightarrow x+y\notin C$.*

2. Existence results

In this section, we establish some existence results for (EVVLI-I) and (EVVLI-II) by using Lemma 1.2.

Lemma 2.1. *Let $S,T: K\to L(X,Y)$, $\eta: K\times K\to X$, $h: K\times K\to Y$ and $g: K\to K$ are the mappings satisfying the following conditions:*

(a) *S and T are g–h–η-pseudomonotone;*
(b) *for any fixed $y\in X$, the mapping $y\mapsto\langle S(y)+T(y),\eta(g(x),g(y))\rangle$ is hemicontinuous and $h(g(x),g(y))$ is continuous with $\{z_t\}\to x_0\in K, z_t\in K$;*
(c) *$h(\cdot,g(y))$ is C-convex in the first variable and $h(g(x),g(x))=0$, $\forall x\in K$;*
(d) *$\eta(\cdot,g(y))$ is affine in the first variable and $\eta(g(x),g(x))=0$, $\forall x\in K$.*

Then for any $x_0\in K$, the following statements are equivalent

(*I*) $\langle S(x_0)+T(x_0),\eta(g(x),g(x_0))\rangle+h(g(x),g(x_0))\geqslant_C 0,\ \forall x\in K,$

(*II*) $\langle S(x)+T(x),\eta(g(x_0),g(x))\rangle+h(g(x_0),g(x))\leqslant_C 0,\ \forall x\in K.$

Proof. As S and T are g–h–η-pseudomonotone, it follows that $(a)\Rightarrow(b)$. Suppose that (b) holds for any $x_0\in K$

$$\langle S(x)+T(x),\eta(g(x_0),g(x))\rangle+h(g(x_0),g(x))\leqslant_C 0,\quad\forall x\in K. \tag{2.1}$$

For arbitrary $z\in K$, letting $z_t=(1-t)x_0+tz, t\in(0,1)$, we have $z_t\in K$ by convexity of K. Hence we have

$$\langle S(z_t)+T(z_t),\eta(g(x_0),g(z_t))\rangle+h(g(x_0),g(z_t))\leqslant_C 0,\quad\forall x\in K. \tag{2.2}$$

Now we show that

$$\langle S(z_t)+T(z_t),\eta(g(z),g(z_t))\rangle+h(g(z),g(z_t))\geqslant_C 0. \tag{2.3}$$

Suppose to the contrary that there exists some $t\in(0,1)$ such that

$$\langle S(z_t)+T(z_t),\eta(g(z),g(z_t))\rangle+h(g(z),g(z_t))\ngeqq_C 0. \tag{2.4}$$

As C is a convex cone and in veiw of $(c),(d)$ we get

$$\begin{aligned}0&=\langle S(z_t)+T(z_t),\eta(g(z_t),g(z_t))\rangle+h(g(z_t),g(z_t))\\ &=\langle S(z_t)+T(z_t),\eta((1-t)g(x_0)+tg(z),g(z_t))\rangle\\ &\quad+h((1-t)g(x_0)+tg(z),g(z_t))\\ &=t\{\langle S(z_t)+T(z_t),\eta(g(z),g(z_t))\rangle+h(g(z),g(z_t))\}\\ &\quad+(1-t)\{\langle S(z_t)+T(z_t),\eta(g(x_0),g(z_t))\rangle+h(g(x_0),g(z_t))\}\\ &\in t\{\langle S(z_t)+T(z_t),\eta(g(z),g(z_t))\rangle+h(g(z),g(z_t))\}\\ &\quad+(1-t)\{\langle S(z_t)+T(z_t),\eta(g(x_0),g(z_t))\rangle+h(g(x_0),g(z_t))\}-C,\end{aligned}$$

which implies that

$$t\{\langle S(z_t)+T(z_t),\eta(g(z),g(z_t))\rangle+h(g(z),g(z_t))\}+(1-t)\\ \times\{\langle S(z_t)+T(z_t),\eta(g(x_0),g(z_t))\rangle+h(g(x_0),g(z_t))\}\\ \notin C.$$

Which is a contradiction. Hence

$$\langle S(z_t)+T(z_t),\eta(g(z),g(z_t))\rangle+h(g(z),g(z_t))\geqq_C 0.$$

Condition (b) implies that

$$\langle S(x_0)+T(x_0),\eta(g(x),g(x_0))\rangle+h(g(x),g(x_0))\geqslant q_C 0,\quad \forall x\in K.$$

This complete the proof. □

Theorem 2.1. *Let X and Y be two real Banach spaces and $K\subset X$ a nonempty, compact and convex set. Let $S,T: K\to L(X,Y)$, $\eta: K\times K\to X, h: K\times K\to Y$ and $g: K\to K$ are the mappings satisfying the following conditions:*

(a) *for any fixed $y\in X$, the mapping $x\mapsto\langle S(x)+T(x),\eta(g(y),g(x))\rangle$ and $h(\cdot,g(x))$ are continuous;*
(b) *S and T are properly g–h–η-quasimonotone of Stampacchia type;*
(c) *for all $x\in K,\eta(g(x),g(x))=0=h(g(x),g(x))$.*

Then there exists $x\in K$ such that

$$\langle S(x)+T(x),\eta(g(y),g(x))\rangle+h(g(y),g(x))\geqslant q_C 0,\quad \forall y\in K.$$

Proof. Define a multivalued mapping $M_1: K\to 2^K$ by

$$M_1(z)=\{x\in K: \langle S(x)+T(x),\eta(g(z),g(x))\rangle\\ +h(g(z),g(x))\geqslant q_C 0\},\ \forall z\in K$$

then $M_1(z)$ is nonempty for each $z\in K$. We claim that M_1 is a KKM mapping. In fact if it is not the case then there exists $\{x_1,x_2,\ldots,x_n\}\subset K, x=\sum_{i=1}^n t_ix_i$ with $t_i>0, i=1,2,\ldots,n$ and $\sum_{i=1}^n t_i=1$ such that $x\notin\bigcup_{i=1}^n M_1(x_i)$.

This implies that

$$\langle S(x)+T(x),\eta(g(x_i),g(x))\rangle+h(g(x_i),g(x))\ngeq_C 0,\ i=1,2,\ldots,n.$$

This contradicts condition (b). Therefore M_1 is a KKM mapping; Now we prove that for any $z\in K, M_1(z)$ is closed.

In veiw of (a), let there exists a net $\{x_n\}\subset M_1(z)$ such that $x_n\to x\in K$. Because

$$\langle S(x)+T(x),\eta(g(z),g(x_n))\rangle+h(g(z),g(x_n))\geqq_C 0,\quad \forall n,$$

we have

$$\langle S(x)+T(x),\eta(g(z),g(x))\rangle+h(g(z),g(x))\geqq_C 0.$$

Hence $x\in M_1(z)$ and so $M_1(z)$ is closed. It follows from the compactness of K and closedness of $M_1(z)\subset K$, that $M_1(z)$ is compact. Thus by Lemma 1.2, we have

$$\bigcap_{z\in K}M_1(z)\neq\phi.$$

Hence there exist $x\in K$ such that

$$\langle S(x)+T(x),\eta(g(y),g(x))\rangle+h(g(y),g(x))\geqq_C 0,\quad \forall y\in K.$$

This complete the proof. □

Theorem 2.2. *Let K be a nonempty, bounded, closed and convex subset a real reflexive Banach space X and Y a real Banach space. Let $S,T: K\to L(X,Y),\eta: K\times K\to X,h: K\times K\to Y$ and $g: K\to K$ are the mappings satisfying the following conditions:*

(a) *S and T are properly g–h–η-quasimonotone of Minty type;*
(b) *$\forall x\in K,\ \eta(g(x),g(x))=0=h(g(x),g(x))=0$. Then there exists $x\in K$ such that*

$$\langle S(y)+T(y),\eta(g(x),g(y))\rangle+h(g(x),g(y))\leqq_C 0,\quad \forall y\in K.$$

Proof. Define a multivalued mapping $M_2: K\to 2^K$ by

$$M_2(z)=\{x\in K: \langle S(z)+T(z),\eta(g(x),g(z))\rangle\\ +h(g(x),g(z))\leqq_C 0\},\quad \forall z\in K$$

then $M_2(z)$ is nonempty for each $z\in K$. Suppose that M_2 is not a KKM mapping, then there exists $\{x_1,x_2,\ldots,x_n\}\subset K$, $x=\sum_{i=1}^n t_ix_i$ with $t_i>0, i=1,2,\ldots,n$ and $\sum_{i=1}^n t_i=1$ such that $x\notin\bigcup_{i=1}^n M_2(x_i)$.

This implies that

$$\langle S(x_i)+T(x_i),\eta(g(x),g(x_i))\rangle+h(g(x),g(x_i))\nleq_C 0,\\ i=1,2,\ldots,n.$$

This contradicts condition (a). Therefore M_2 is a KKM mapping. In addition, it is easy to verify that $M_2(z)$ is bounded, closed and convex for all $z\in K$. Since X is reflexive, $M_2(z)$ is weakly compact for all $z\in K$. It follows from Lemma 1.2 that

$$\bigcap_{z\in K}M_2(z)\neq\phi.$$

Hence there exists $x\in K$ such that

$$\langle S(y)+T(y),\eta(g(x),g(y))\rangle+h(g(x),g(y))\leqq_C 0,\quad \forall y\in K.$$

This completes the proof. □

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